ON THE REALIZATION OF THE DISCRETE SERIES OF A SEMISIMPLE LIE GROUP

*by Wilfried Schmid**

Let G be a connected, semisimple Lie group, and $K \subset G$ a maximal compact subgroup. I shall assume that G and K are of equal rank. Thus one can choose a Cartan subgroup H of G which is contained in K. It is known that G/H carries various G-invariant complex structures, and that each character of H determines a homogeneous, holomorphic line bundle on G/H. During the Boulder Conference, R. P. Langlands [5] suggested that the representations of Harish-Chandra's discrete series [3] might be realized on L^2-cohomology groups of such line bundles, in analogy to the generalized Borel-Weil theorem for compact groups. Since then, M. S. Narasimhan and Okamoto have proven the corresponding statement about vector bundles over G/K, provided this symmetric space has an invariant complex structure [6], and Langlands' conjecture for G/H has also been verified [8] (in both cases, however, only discrete series representations with "sufficiently non singular" infinitesimal character are constructed, because of certain technical difficulties).

In some ways, it is simpler to work on G/K instead of G/H, and one may ask if the discrete series representations of G can be obtained as solution spaces of differential operators on G/K, even if G/K does not admit an invariant complex structure. In [7], I mentioned a conjecture to this effect. Each irreducible K-module V determines a homogeneous vector bundle $V \to G/K$; corresponding to every choice of positive roots, there is defined a certain homogeneous, first order differential operator $\mathscr{D}$ on V. If G/K is hermitian symmetric and the positive root system is suitably chosen, $\mathscr{D}$ reduces to the Cauchy-Riemann operator $\bar{\partial}$. In any case, under suitable hypotheses on V, $\mathscr{D}$ is elliptic. The space of square-integrable sections of V in the kernel of $\mathscr{D}$ thus becomes a Hilbert space, on which G acts unitarily. As was suggested in [7], it should be possible to realize the discrete series representations on these Hilbert spaces. Several properties

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of the discrete series representations would be a consequence, because the space of all C^∞ solutions of $\mathscr{D}F = 0$ is isomorphic to a cohomology group of a homogeneous, holomorphic line bundle over G/H and has been studied with the methods of complex analysis [7].

In this note I shall define the operator $\mathscr{D}$, give a quick proof of its ellipticity, and sketch an argument showing that "most" discrete series representations arise as L^2 kernels of $\mathscr{D}$. This last fact has also been established by Harish-Chandra. Some of the arguments below are similar to those in [8], but simpler. It seems possible that they can be refined to give all of the discrete series of at least those groups which have a faithful finite-dimensional representation. Finally I shall mention—without proof—some consequences, such as an analogue of the theorem of the highest weight, as well as a partial result towards a conjecture of Blattner. Full details will appear elsewhere.

I denote the Lie algebras of G, K, H by $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{h}_0$, and their complexifications by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$. Since G/K is Riemannian symmetric, there exists a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p};$$

$\mathfrak{p}$ is the complexification of $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0$. The differentials of the characters of the torus H form a lattice Λ in $\mathfrak{h}_{\boldsymbol{R}}{}^*$, the dual space of

$$\mathfrak{h}_{\boldsymbol{R}} = \sqrt{-1}\,\mathfrak{h}_0.$$

Let $\Delta \subset \Lambda$ be the set of non zero roots of $(\mathfrak{g}, \mathfrak{h})$, and

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$$

the rootspace decomposition. A root $\alpha \in \Delta$ is said to be compact or non compact according to whether $\mathfrak{g}^\alpha \subset \mathfrak{k}$ or $\mathfrak{g}^\alpha \subset \mathfrak{p}$; the sets of compact and non compact roots will be referred to as Δ_K and Δ_P, respectively. Suppose now that a system of positive roots $\Delta^+ \subset \Delta$ is given; Δ^+ will not be kept fixed, but rather will be allowed to vary. For any given choice of Δ^+, I define

$$\Delta_K^+ = \Delta_K \cap \Delta^+, \quad \Delta_P^+ = \Delta_P \cap \Delta^+,$$

$$\rho_K = \tfrac{1}{2} \textstyle\sum_{\Delta_K^+} \alpha, \; \rho_P = \tfrac{1}{2} \sum_{\Delta_P^+} \alpha,$$

$$\Lambda^+ = \{\lambda \in \Lambda \,|\, (\lambda, \alpha) \geqq 0 \text{ for all } \alpha \in \Delta_K^+\}.$$

The inner product on $\mathfrak{h}_R^*$ used to define Λ^+ is, of course, the one induced by the Killing form. Thus Λ^+ consists precisely of the weights which are dominant with respect to the system of positive roots Δ_K^+ of $(\mathfrak{k}, \mathfrak{h})$.

Every $\lambda \in \Lambda^+$ is the highest weight of an irreducible K-module V_λ. If π_1, π_2 are two representations of a complex, reductive Lie algebra, and π_1 is irreducible, then the highest weight of any irreducible component of $\pi_1 \otimes \pi_2$ differs from the highest weight of π_1 by some weight of π_2. Thus the highest weights occurring in the K-module $V_\lambda \otimes \mathfrak{p}$ are all of the form $\lambda + \beta$, $\beta \in \Delta_P$. I shall denote the direct sum of the components with highest weight of the form $\lambda - \beta$, $\beta \in \Delta_P^+$, by U_λ, and the K-invariant projection of $V_\lambda \otimes \mathfrak{p}$ onto U_λ by

$$q: V_\lambda \otimes \mathfrak{p} \to U_\lambda .$$

The K-modules V_λ, U_λ associate vector bundles

$$\boldsymbol{V}_\lambda \to G/K, \quad \boldsymbol{U}_\lambda \to G/K$$

to the principal bundle

$$K \to G \to G/K .$$

The space of C^∞ sections of $\boldsymbol{V}_\lambda, \Gamma(\boldsymbol{V}_\lambda)$, can be identified naturally with

$$\{C^\infty(G) \otimes V_\lambda\}_K,$$

i.e., the space of K-invariant elements of $C^\infty(G) \otimes V_\lambda$ when K is made to act on $C^\infty(G)$ by right translation, and on V_λ in the obvious manner. Similarly

$$\Gamma(\boldsymbol{U}_\lambda) \simeq \{C^\infty(G) \otimes U_\lambda\}_K .$$

Via left translation, G acts on $\Gamma(\boldsymbol{V}_\lambda)$ and $\Gamma(\boldsymbol{U}_\lambda)$. Since the Killing form B of $\mathfrak{g}$ restricts to a non degenerate, bilinear, symmetric form on $\mathfrak{p}$, one can choose a basis $\{x_i\}$ of $\mathfrak{p}$ satisfying

$$B(x_i, x_j) = \delta_{ij} . \tag{1}$$

The x_i may be regarded as left-invariant, complex tangent vector fields on G. Then, whenever F is an element of $\{C^\infty(G) \otimes V_\lambda\}_K$,

$$\sum_i q(x_i F \otimes x_i)$$

belongs to $\{C^\infty(G) \otimes U_\lambda\}_K$. Hence the map

$$F \to \sum_i q(x_i F \otimes x_i) \tag{2}$$

determines a first order differential operator

$$\mathscr{D}:\Gamma(V_\lambda) \to \Gamma(U_\lambda),$$

which evidently commutes wth the action of G; it depends on the choice of positive roots, but not on the particular basis $\{x_i\}$. In [7] I stated that $\mathscr{D}$ is elliptic for every "very positive" $\lambda \in \Lambda$; Hotta [4] has obtained the ellipticity under weaker conditions. These results can be sharpened to:

Lemma 1. *The operator $\mathscr{D}$ is elliptic, provided*

$$\lambda - 2\rho_P \in \Lambda^+ . \tag{3}$$

Proof. Because of the G-invariance, it suffices to consider the operator at eK. The real tangent space of G/K at eK is naturally isomorphic to $\mathfrak{p}_0$, and by means of the Killing form, $\mathfrak{p}_0$ also can be identified with the real cotangent space. With this identification, the symbol mapping of $\mathscr{D}$ at eK becomes

$$\sigma(\mathscr{D},\xi): v \to q(v \otimes \xi), \quad v \in V_\lambda,\ \xi \in \mathfrak{p}_0 .$$

This map must be shown to be injective if $\xi \neq 0$. The condition (3) ensures the existence of an irreducible K-module W with the highest weight $\lambda - 2\rho_P$. Let $2n$ be the complex dimension of $\mathfrak{p}$. Since $2\rho_P$ is just the sum of all positive, non compact roots, one can choose a K-invariant embedding $V_\lambda \xrightarrow{\phi} W \otimes \Lambda^n \mathfrak{p}$. For $x \in \mathfrak{p}$, let

$$e(x): \Lambda\mathfrak{p} \to \Lambda\mathfrak{p}$$

be exterior multiplication, and

$$i(x): \Lambda\mathfrak{p} \to \Lambda\mathfrak{p}$$

the adjoint operation, relative to the inner product

$$(x,y) = B(x,\bar{y}) \tag{4}$$

on $\mathfrak{p}$ (barring denotes conjugation with respect to $\mathfrak{g}_0$). Then $v \otimes x \to e(x)\phi(v)$ and $v \otimes x \to i(\bar{x})\phi(v)$ are K-invariant homomorphisms of $V \otimes \mathfrak{p}$ intc $W \otimes \Lambda^{n+1}\mathfrak{p}$ and $W \otimes \Lambda^{n-1}\mathfrak{p}$, respectively. Since these K-modules do not contain irreducible components with highest weights of the form $\lambda + \beta$, $\beta \in \Delta_P^+$, $q(v \otimes \xi) = 0$ implies that $e(\xi)\phi(v) = 0$ and $i(\xi)\phi(v) = 0$. On the other hand,

$$e(\xi)i(\xi) + i(\xi)e(\xi)$$

is multiplication by (ξ,ξ). Thus $\sigma(\mathscr{D},\xi) = 0$ only if $\xi = 0$.

In view of the proposition, one might suspect that $\mathscr{D}$ can be resolved by a G-invariant, elliptic complex. For a particular choice of the positive root system, Hotta has constructed such a resolution [*4*].

Suppose now that $\lambda \in \Lambda^+$ satisfies (3). Since K is compact and operates irreducibly on V_λ, the vector bundle $\mathbf{V}_\lambda$ admits an essentially unique hermitian metric; similarly, G/K carries an essentially unique volume form dv. Let $H_\lambda \subset \Gamma(\mathbf{V}_\lambda)$ be the space of square-integrable solutions of $\mathscr{D}F = 0$. Because of the regularity theorem for elliptic operators, H_λ is a Hilbert space under its natural inner product. Clearly G acts unitarily on H_λ.

I shall say that a statement involving an element λ of Λ is true for every sufficiently non singular λ, provided there exists a constant c such that the condition

$$\left|(\lambda, \alpha)\right| > c \text{ for all } \alpha \in \Delta$$

ensures the correctness of the statement.

Theorem 1. *If the system of positive roots Δ^+ and $\lambda \in \Lambda$ are such that*

$$(\lambda, \alpha) \geqq 0 \text{ for all } \alpha \in \Delta^+, \tag{5}$$

and if λ is sufficiently non singular (in particular, (3) should hold), then the Hilbert space H_λ is non zero, G acts irreducibly, the resulting representation belongs to the discrete series, and its character is $\Theta_{\lambda - \rho_P + \rho_K}$ (*cf.* [3]).

If it were possible to relax the hypothesis by requiring only (3), and not that λ is sufficiently non singular, the theorem would give a realization of all the discrete series for groups with a faithful finite-dimensional representation. Before outlining a proof of Theorem 1, I shall state some facts which follow more or less directly from it, together with the results of [*7*].

Let W be the Weyl group of $(\mathfrak{k}, \mathfrak{h})$, and ε_w, for $w \in W$, the sign of w. I define $Q(\mu)$, $\mu \in \Lambda$, as the number of distinct ways in which μ can be expressed as a (possibly empty) sum of positive, non compact roots. Blattner has conjectured that for every $\mu \in \Lambda^+$,

$$k_\lambda(\mu) = \sum_{w \in W} \varepsilon_w Q(w(\mu + \rho_K) - (\lambda + \rho_K))$$

represents the multiplicity of the irreducible K-module with highest weight μ

in the discrete series representation whose character is $\Theta_{\lambda-\rho_P+\rho_K}$. Formally, this resembles Kostant's formula for the multiplicity of a weight in the finite-dimensional case.

Theorem 2. *Under the hypotheses of Theorem* 1, *among all irreducible, unitary representations of* G, *the discrete series representation with character* $\Theta_{\lambda-\rho_P+\rho_K}$ *is characterized uniquely, up to equivalence, by any one of the following conditions*:

1) *It contains the irreducible* K-*module of highest weight* λ, *but not those with highest weights of the form* $\lambda-\beta$, $\beta\in\Delta_P^+$.

2) *It contains the irreducible* K-*module with highest weight* λ, *and the Casimir operator acts as multiplication by a constant greater than or equal to* $(\lambda+2\rho_K,\lambda-2\rho_P)$.

3) *Its character has the same restriction to* K *as* $\Theta_{\lambda-\rho_P+\rho_K}$.

Moreover, the irreducible K-*module of highest weight* λ *occurs exactly once in this discrete series representation, and the multiplicity of the irreducible* K-*module of highest weight* $\mu\in\Lambda^+$ *is no greater than* $k_\lambda(\mu)$.

The first two statements are analogous to the theorem of the highest weight about finite-dimensional representations, and the fact that among all irreducible, finite-dimensional representations which involve a particular weight, the Casimir operator is minimized by the one for which the given weight is extreme. The last assertion gives at least a part of Blattner's conjecture. It seems quite possible that the opposite inequality can now be obtained by combinatorial arguments. For example, such arguments easily imply

Proposition 1. *Suppose the assumptions of Theorem* 1 *are satisfied. If the system of positive roots* Δ^+ *has the property that there exists exactly one non compact, simple root whose coefficient in the highest root is one or two, then Blattner's conjecture holds for the representation of* G *on* H_λ.

According to Borel and de Siebenthal [1], every non compact, simple Lie group G, with rank $G=\operatorname{rank} K$, has a positive root system of this kind. If the coefficient of the simple, non compact root in the highest root is only one, G/K admits an invariant complex structure, and the representations mentioned in Theorem 1, corresponding to this special positive root system, were constructed by Harish-Chandra [2]; in this case, Blattner's conjecture is well understood.

In order to attack the proof of Theorem 1, I consider an arbitrary irreducible representation π of G on a Hilbert space H, and I denote the subspace of vectors which transform finitely under K by H_0. The Lie algebra $\mathfrak{g}$ operates on H_0, and this action will also be referred to as π. The adjoint action of K on $\mathfrak{p}$ induces an action ψ on the exterior algebra $\wedge\mathfrak{p}$. For $x \in \mathfrak{p}$ I define the endomorphisms $e(x)$, $i(x)$ as in the proof of Lemma 1. Choose a basis $\{x_i\}$ of $\mathfrak{p}$ which satisfies (1). Then

$$d = \sum_i \pi(x_i) \otimes e(x_i)$$

defines a K-invariant linear transformation

$$d : H_0 \otimes \wedge\mathfrak{p} \to H_0 \otimes \wedge\mathfrak{p},$$

whose formal adjoint is

$$d^* = -\sum_i \pi(\bar{x}_i) \otimes i(x_i).$$

A straightforward computation, which will be omitted, gives

Lemma 2. *Let Ω be the Casimir operator of G, and Ω_K the Casimir operator of K. Then*

$$\begin{aligned} d^*d + dd^* &= \tfrac{1}{2}\,\pi(\Omega_K) \otimes 1 + \tfrac{1}{2}(\pi \otimes \psi)(\Omega_K) \\ &\quad - \tfrac{1}{2}\,1 \otimes \psi(\Omega_K) - \pi(\Omega) \otimes 1, \\ d^2 &= \tfrac{1}{2} \sum_{i,j} \pi([x_i, x_j]) \otimes e(x_i)e(x_j), \quad \text{and} \\ d^{*2} &= \tfrac{1}{2} \sum_{i,j} \pi([\bar{x}_i, \bar{x}_j)] \otimes i(x_i)i(x_j). \end{aligned}$$

Evidently it is possible to embed the irreducible K-module $V^*_{2\rho_P}$, whose lowest weight is $-2\rho_P$, in $\wedge\mathfrak{p}$. If the irreducible K-module V_λ occurs in H_0, and if λ satisfies (3), the subspace

$$V_\lambda \otimes V^*_{2\rho_P} \subset H_0 \otimes \wedge\mathfrak{p}$$

contains vectors which transform under $\pi \otimes \psi(K)$ according to the highest weight $\lambda - 2\rho_P$. By applying the first identity of Lemma 1 to such a vector and noticing that $d^*d + dd^*$ is positive semidefinite, one obtains $\pi(\Omega) \leqq (\lambda + 2\rho_K, \lambda - 2\rho_P)$.

Thus:

Lemma 3. *If π contains the irreducible K-module of highest weight λ, with λ satisfying* (3), *the Casimir operator acts as multiplication by a constant not exceeding $(\lambda + 2\rho_K, \lambda - 2\rho_P)$.*

Each irreducible K-module occurs only finitely often in $H_0 \otimes \wedge \mathfrak{p}$, and the formally self-adjoint operators $d^*d + dd^*$ and $d^2 + d^{*2}$ commute with the action of K. Hence these operators can be diagonalized. As a consequence of Lemma 2, the eigenvalues of $d^*d + dd^*$ grow approximately at the same rate as those of $\pi(\Omega_K) \otimes 1$; on the other hand, since $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, the eigenvalues of $d^2 + d^{*2}$ can be bounded approximately by the square-root of those of $\pi(\Omega_K) \otimes 1$. The eigenvalues of $(d + d^*)^2$ therefore tend to ∞ outside of finite-dimensional subspaces. It follows that both $(d + d^*)^2$ and $d + d^*$ have finite-dimensional kernel and cokernel. For $\mu \in \Lambda^+$, let n_μ be the multiplicity in π of the irreducible K-module of highest weight μ and χ_μ its character. Then $\sum_\mu n_\mu \chi_\mu$ is the Fourier series of a distribution τ on K, which is real-analytic on K intersected with the regular set, where it agrees with the character of π. Being invariant under the Weyl group W, the expression

$$D = \prod_{\beta \in \Delta_P^+} (e^{\beta/2} - e^{-\beta/2})$$

extends to a class function on K. The grading of $\wedge\mathfrak{p}$, reduced mod 2, gives a splitting

$$\wedge\mathfrak{p} = \wedge^+\mathfrak{p} \otimes \wedge^-\mathfrak{p}.$$

Evidently $D^2\tau$ agrees, up to sign, with the difference of the characters of these two K-modules. In a formal sense, $D^2\tau$ is therefore the difference of the characters of $H_0 \otimes \wedge^+\mathfrak{p}$ and $H_0 \otimes \wedge^-\mathfrak{p}$. But because

$$d + d^*: H_0 \otimes \wedge^+\mathfrak{p} \to H_0 \otimes \wedge^-\mathfrak{p}$$

has finite-dimensional kernel and cokernel, as was pointed out above, that difference of characters must be a finite integral linear combination of irreducible characters. This proves

Lemma 4. *The distribution $D^2\tau$ is a function.*

Actually, even $D\tau$ is a function. When G/K admits an invariant complex structure, one can prove this by considering a K-invariant splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_+ \oplus \overline{\mathfrak{p}_+}$ and working with $\wedge\mathfrak{p}_+$ instead of $\wedge\mathfrak{p}$. If G/K is not hermitian symmetric, the proof becomes more delicate; it can be found in [*8*].

Suppose now that π is the discrete series representation with character $\Theta_{\lambda-\rho_P+\rho_K}$. In view of Lemma 4, $D^2\tau$ can be read off from Harish-Chandra's character formula, and one finds:

the K-module with highest weight $\lambda - 2\rho_P$ occurs in $H_0 \otimes \wedge \mathfrak{p}$, which is equivalent to:

(6) some irreducible component of $V_{\lambda - 2\rho_P} \otimes \wedge \mathfrak{p}$ occurs in H_0.

If λ has a large inner product with every positive, compact root, every highest weight μ in $V_{\lambda - 2\rho_P} \otimes \Lambda \mathfrak{p}$ satisfies the condition

$$(\mu - 2\rho_P, \alpha) \geqq 0 \quad \text{for} \quad \alpha \in \Delta_K^+ ,$$

and $\mu = \lambda$ is the only highest weight in $V_{\lambda - 2\rho_P} \otimes \wedge \mathfrak{p}$ with

$$(\mu + 2\rho_K, \mu - 2\rho_P) \geqq (\lambda + 2\rho_K, \lambda - 2\rho_P).$$

Since the Casimir operator acts as multiplication by the constant on the right, (6) and Lemma 3 imply that H_0 contains the K-module V_λ; and it contains it exactly once because of the explicit form of the character formula. As can be shown by a similar application of Lemma 3, if λ is sufficiently non singular, no irreducible K-module with a highest weight of the form $\lambda - \beta$, $\beta \in \Delta_P^+$, can occur in π. This establishes the next lemma, which has also been obtained by Harish-Chandra.

Lemma 5. *Suppose λ satisfies (5) and is sufficiently non singular. The discrete series representation with character $\Theta_{\lambda - \rho_P + \rho_K}$ contains the irreducible K-module V_λ exactly once, but no irreducible K-module with highest weight of the form $\lambda - \beta, \beta \in \Delta_P^+$.*

Let $\phi: V_\lambda \to H$ be a particular K-invariant embedding of V_λ in the representation space of this discrete series representation, and let $\{v_i\}$ be an orthonormal basis of V_λ. Every vector $u \in H$ gives rise to a square-integrable, V_λ-valued function

$$F_u : g \to \sum_i (\pi(g^{-1})u, \phi(v_i))v_i$$

on G. Since H contains no irreducible component of U_λ,

$$\sum_i q((x_i F_u)(g) \otimes v_i) = \sum_{i,j} (\pi(g^{-1})u, \pi(x_i)\phi(v_j))q(v_j \otimes x_i)$$

(cf. (2)) must vanish identically. Thus $u \to F_u$ defines a G-invariant embedding of H in H_λ. The multiplicity of V_λ in H is one, so that H occurs just once in $\{L^2(G) \otimes V_\lambda\}_K$, and therefore certainly no more than once in H_λ.

For any other irreducible unitary representation which is contained or weakly contained in H_λ, the Casimir operator must act also as multiplication by $(\lambda + 2\rho_K, \lambda - 2\rho_P)$; one can show this by combining the idea of the proof of Lemma 1 with the first identity of Lemma 2. Reversing the argument leading to Lemma 5, one would find that $D^2\tau$ involves the irreducible character $\chi_{\lambda-2\rho_P}$ with non zero coefficient, which excludes the occurrence in H_λ of all but one discrete series representation. It follows from Harish-Chandra's results that the set of irreducible, unitary representations, outside of the discrete series, on which the Casimir operator acts as multiplication by a fixed constant, has Plancherel measure zero; hence only the discrete series can contribute to H_λ. This fact concerning the Plancherel measure is, of course, quite deep. Instead, one may appeal to the results of [7], which imply directly that H_λ, unless it is zero, must be irreducible.

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